JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 192, 667–668 (1995)

Note

The Completion of a Pseudo-Barrelled Space Is Pseudo-Barrelled

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Submitted by John Horváth

Received March 21, 1994

We follow [2] for notation and terminology and also assume that all spaces under discussion are separated. Recall that a locally convex space in which every lower semicontinuous seminorm is continuous is called barrelled, while one in which every lower semicontinuous seminorm bounded on bounded subsets is continuous is called infrabarrelled. Gorniak in [1] introduced the class of pseudo-barrelled spaces; a space is called pseudo-barrelled if every lower semicontinuous seminorm that is continuous on the bounded subsets is continuous. There he gave some properties, an example, and subclasses of pseudo-barrelled spaces. For instance, he stated (without proof) that an inductive limit of pseudo-barrelled spaces is pseudo-barrelled; from this we may deduce immediately that a separated quotient of a pseudo-barrelled space is pseudo-barrelled.

This paper adds the following property:

THEOREM. *The completion of a pseudo-barrelled space is pseudo-barrelled.*

Denote by $(\hat{E}, \hat{T})$ the completion of the separated locally convex space (E, T) and by $\hat{q}$ the unique extension of the seminorm q on (E, T) to the completion $(\hat{E}, \hat{T})$.

LEMMA 1. [2, p. 134]. *If the topology T of the locally convex space (E, T) is defined by the collection of seminorms $\{q_i\}_{i \in I}$, then the collection $\{\hat{q}_i\}_{i \in I}$ of their extensions to the completion $(\hat{E}, \hat{T})$ of (E, T) defines the topology $\hat{T}$.*

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LEMMA 2. [2, Exercise 3.6.1, p. 219]. *Seminorm p on (E, T) is lower semicontinuous* $\leftrightarrow$ $\{x \in E : p(x) \leq 1\}$ *is a barrel.*

LEMMA 3. *If the seminorm p on $(\hat{E}, \hat{T})$ is lower semicontinuous so is its restriction $p|_E$ to (E, T).*

Proof. If $\hat{B} = \{x \in \hat{E}: p(x) \leq 1\}$, then clearly $\hat{B} \cap E = \{x \in E : p|_E(x) \leq 1\}$. By Lemma 2 $\hat{B}$ is a barrel in $(\hat{E}, \hat{T})$: $\hat{B} \cap E$ is absolutely convex and closed in (E, T), $\hat{B}$ being closed in $(\hat{E}, \hat{T})$. We conclude the proof by showing that $\hat{B} \cap E$ is absorbing. Let $x \in E$. Since $\hat{B}$ is absorbing we have $x \in t\,\hat{B}$ for some $t > 0$. Hence $t^{-1} x \in \hat{B}$. E being a vector space $t^{-1} x \in E$, so $t^{-1} x \in \hat{B} \cap E$. Hence $x \in t\,(\hat{B} \cap E)$. Then by [2, the first two lines after Definition 2.3.3 on p. 80] we have that $\hat{B} \cap E$ is absorbing. ∎

LEMMA 4. *Bounded subsets of (E, T) remain bounded in $(\hat{E}, \hat{T})$.*

Proof. The proof is immediate from the identification of a fundamental system of neighbourhoods of zero in $(\hat{E}, \hat{T})$ given in [2, p. 134]. ∎

Proof of the Theorem. Let p be a lower semicontinuous seminorm on $(\hat{E}, \hat{T})$ that is continuous on bounded sets of $(\hat{E}, \hat{T})$. Hence by Lemma 3, $p|_E$ is lower semicontinuous on (E, T) and by Lemma 4 it is continuous on bounded subsets of (E, T). (E, T) being pseudo-barrelled implies that $p|_E$ is continuous on (E, T). Hence, $B_{p|_E} = \{x \in E : p|_E(x) \leq 1\}$ is a neighborhood of zero in (E, T). Now, p being lower semicontinuous,

$$B_p = \{x \in \hat{E} : p(x) \leq 1\}$$

is closed in $(\hat{E}, \hat{T})$, B_p being a barrel in $(\hat{E}, \hat{T})$. Also,

$$B_p \supseteq B_{p|_E},$$

hence

$$B_p \supseteq \overline{B_{p|_E}}^{(\hat{E},\hat{T})},$$

where $\overline{}^{(\hat{E},\hat{T})}$ indicates the closure taken in $(\hat{E}, \hat{T})$. But $\overline{B_{p|_E}}^{(\hat{E},\hat{T})}$ is a neighborhood of zero in $(\hat{E}, \hat{T})$ and so B_p is also, and hence p is continuous. ∎

REFERENCES

1. J. GORNIAK, Locally convex spaces with factorization property, *Colloq. Math.* **48** (1984), 69–79.
2. J. HORVÁTH, "Topological Vector Spaces and Distributions," Vol. 1, Addison-Wesley, Reading, MA, 1966.